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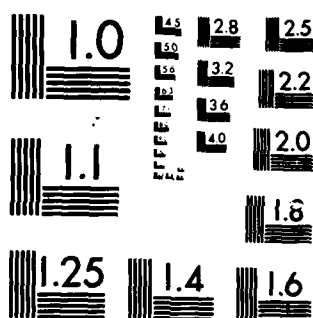
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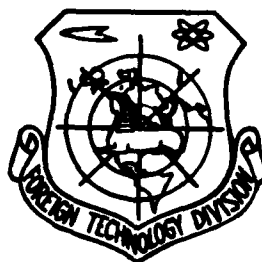
## FOREIGN TECHNOLOGY DIVISION



LIMIT CYCLE OF THE PLANAR NON-LINEAR MOTION OF A FINNED PROJECTILE

by

Han Zipeng



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## EDITED TRANSLATION

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# Limit Cycle of the Planar Non-linear Motion of a Finned Projectile

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Han Zipeng

[Abstract] In this paper, the planar non-linear motion of a finned projectile with small damp and weak non-linear static moment has been studied. The stable limit cycle is used to explain the self-oscillation generated by some projectiles. The unstable limit cycle is used to investigate the causes for the fortuitous fall of some finned projectiles. In order to alleviate the effect of self-oscillation and to eliminate the fortuitous fall of projectiles, the non-linear portion of the damp moment and that of lift should be increased and the non-linear part of the drag should be decreased.

## 1. First Approximation Solution of the Non-linear Equation of Motion of a Finned Projectile

When the non-linear aerodynamic characteristics are considered, the angular equation of motion of a finned projectile is<sup>[1]</sup>

$$\delta'' + 2(b_0 + b_2\delta^2)\delta' - (k_0 + k_2\delta^2)\delta = 0 \quad (1.1)$$

where

$$b_0 = -\frac{1}{2}(k_{m0} + b_{y0} - b_m - \frac{\rho \sin \theta}{g^2}), \quad b_2 = -\frac{1}{2}(k_{m2} + b_{y2} - b_{m2}) \quad (1.2)$$

$$k_{mj} = \frac{\rho S}{2A} d l m'_{ij}(M), \quad k_{yj} = \frac{\rho S}{2A} l m'_{ij}(M), \quad (j = 0, 2) \quad (1.3)$$

$$b_{yj} = \frac{\rho S}{2m} C_{yij}(M), \quad b_{mj} = \frac{\rho S}{2m} C_{mij}(M), \quad (j = 0, 2) \quad (1.4)$$

where  $\delta$ -angle of attack;  $v$ -velocity of the center of mass of the projectile;  $\theta$ -angle of inclination of the trajectory;  $\rho$ -density of air;  $S$ -characteristic area;  $d$ -diameter of the projectile;  $A$ -equatorial rotational inertia;  $l$ -characteristic length;  $c_{x_0}$  and  $C_{x_2}$ -constant and quadratic coefficients of the drag coefficient;  $C_{y_0}$  and  $C_{y_2}$ -linear and cubic coefficients of the lift coefficient;  $m_{z_0}$  and  $m_{z_2}$ -linear and cubic coefficients of the static moment coefficient; and  $m'_{zz_0}$  and  $m'_{zz_2}$ -constant and quadratic coefficient in the equatorial moment coefficient.

Based on various aerodynamic shapes of the projectiles, the position of the center of gravity and the Mach number, the algebraic signs of  $b_0$  and  $b_2$  may have the following four combinations<sup>[6]</sup>:

(1)  $b_0 < 0$ ,  $b_2 > 0$ ; (2)  $b_0 > 0$ ,  $b_2 < 0$ ; (3)  $b_0 > 0$ ,  $b_2 > 0$ ; (4)  $b_0 < 0$ ,  $b_2 < 0$

In the following, we used the van der pol method<sup>[2][3][4]</sup> to study the motion of a finned projectile with small damp ( $|b_0 / \sqrt{k_{z_0}}| \ll 1$ ) and weak non-linear static moment ( $|k_{z_2} \delta^2| \ll |k_{z_0}|$ ). The majority of the projectiles belong to this case.

Let us leave the conservative moment  $k_{z_0} \delta$  in equation (1.1) on the left side and move the remaining moment terms to the right. Equation (1.1) can be written in the following form:

$$\ddot{\delta} + \delta = f(\delta, \dot{\delta}) \quad (1.5)$$

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where

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$$f(\delta, \dot{\delta}) = -\frac{2}{\sqrt{-k_{z_0}}} (b_0 + b_1 \delta^2) \dot{\delta} + \frac{k_{z_1}}{k_{z_0}} \delta^3 \quad (1.6)$$

where " $\delta$ " represents taking the derivative with respect to the variable  $\tau = \omega s$  and  $\omega = \sqrt{-k_{z_0}}$ . If we neglect the interference term  $f(\delta, \dot{\delta})$  of small damp and weak non-linear static moment, then a linear vibration  $\ddot{\delta} + \delta = 0$  can be obtained. Its solution is a simple harmonic motion

$$\delta = R \cos(\tau + \varphi) \quad (1.7)$$

$$\dot{\delta} = -R \sin(\tau + \varphi) \quad (1.8)$$

where

$$R = \sqrt{\delta_0^2 + (\dot{\delta}_0/\omega)^2}, \quad \varphi = \text{tg}^{-1}(-\dot{\delta}_0/\omega\delta_0) \quad (1.9)$$

The amplitude  $R$  and the phase angle  $\varphi$  are determined by the initial conditions  $\delta_0$  and  $\dot{\delta}_0$  (which is the derivative of arc length  $s$  with respect to  $\delta$ ). When the effect of  $f(\delta, \dot{\delta})$  is further considered, it is believed that the solution of equation (1.1) still possesses the form of that in equations (1.7) and (1.8). In this case, however,  $R$  and  $\varphi$  are no longer constants. Instead, they are functions of  $\tau$ .  $\delta$  and  $\dot{\delta}$  can be derived from equation (1.7). Then, equations (1.8) and (1.1) are used to obtain the two equations relating to  $dR/d\tau$  and  $d\varphi/d\tau$ . Based on these equations, we can solve

$$\frac{dR}{d\tau} = -f[R\cos(\tau + \varphi), -R\sin(\tau + \varphi)]\sin(\tau + \varphi) \quad (1.10)$$

$$\frac{d\varphi}{d\tau} = -\frac{1}{R} f[R\cos(\tau + \varphi), -R\sin(\tau + \varphi)]\cos(\tau + \varphi) \quad (1.11)$$

Because  $R$  and  $\varphi$  vary for slower than the variable  $\zeta = \tau + \varphi$ , therefore, in the first approximation the right side of the two above equations can be considered as periodic functions of the variable  $\zeta$ . The period is  $2\pi$ . Thus, the right side of both equations above can be expanded into Fourier series. In addition, the effect of various oscillation terms in the Fourier series can be omitted in the first approximation. Only constant terms remain. It is then substituted into the expression for  $f(\delta, \dot{\delta})$  (i.e., equation (1.6)). The following result is obtained by calculation

$$\frac{dR}{d\tau} = -\frac{1}{2\pi} \int_0^{2\pi} f(R \cos \zeta, -R \sin \zeta) d\zeta = -\frac{R}{4\sqrt{-k_{x0}}} (4b_0 + b_2 R^2) = F(R) \quad (1.12)$$

$$\frac{d\varphi}{d\tau} = -\frac{1}{2\pi R} \int_0^{2\pi} f(R \cos \zeta, -R \sin \zeta) d\zeta = \frac{3k_{x0}}{8k_{x0}} R^2 = \Phi(R) \quad (1.13)$$

## 2. Phase Plane Analysis

In the xoy phase plane, the position of the phase point B is determined by  $x=\delta$  and  $y=\dot{\delta}$ . The motion of the phase point in the phase plane reflects the oscillation of the projectile. Based on equations (1.7) and (1.8), however, the position of the phase point B can also be determined by the phase radius  $R$  and phase angle  $\tau + \varphi$  ( $\tau$ ) (See Figure 2.1).

Based on van der pol's method, a moving phase plane  $o\xi\eta$  is chosen, which rotates clockwise at an angular velocity  $\omega$ . In this moving phase plane, the polar coordinates of the phase point B happen to be  $R(\tau)$  and  $\varphi(\tau)$ . Therefore, equations (1.12) and

(1.13) describe the pattern of the motion of the phase point in the moving phase plane.

Let us first discuss the case of linear static moment where  $k_{z_2} = 0$ ,  $\phi(R) = 0$  and  $\varphi = \varphi_0$ . Hence, singular points are located along a line at an azimuth  $\varphi$ . The phase radius of the singular point is determined by  $F(R) = 0$ .

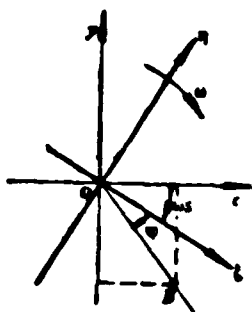


Figure 2.1 Dynamic Phase Plane and Fixed Phase Plane

(1) When  $b_0 < 0$ ,  $b_2 > 0$ , we know from Equations (1.12) and (1.13) that there are two singular points in the dynamic phase plane; i.e.,  $(0,0)$  and  $(R_p = 2\sqrt{-b_0/b_2}, \varphi_0)$ . When  $0 < R < R_p$ ,  $\dot{R} > 0$  and /64 the radius of the phase point continues to increase. When  $R > R_p$ ,  $\dot{R} < 0$  and the phase radius continues to decrease. Only at  $R = R_p$ ,  $\dot{R} = 0$  and the phase radius remain unchanged. Therefore, the second singularity is a stable nodal point in the dynamic phase plane. Correspondingly, this singular point draws a circle with a radius  $R^p$  in the fixed phase plane. When the phase point approaches this singular point radially along  $\varphi_0$  in the dynamic

phase plane, a spiral can be drawn in the fixed phase plane. Hence, the circle is the stable limit cycle in the fixed phase plane. In this case, the origin is an unstable focus. (See Figure 2.2(a)).

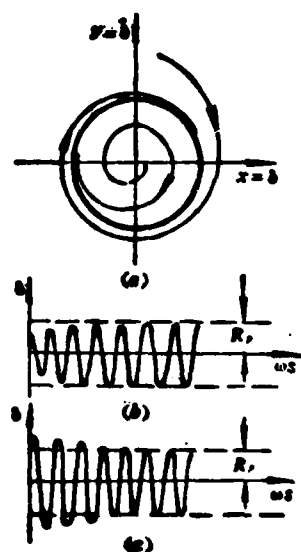


Figure 2.2 A Stable Limit Cycle

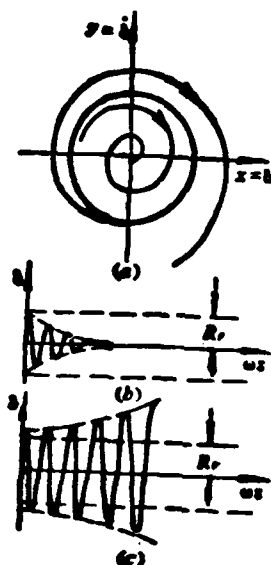


Figure 2.3 An Unstable Limit Cycle

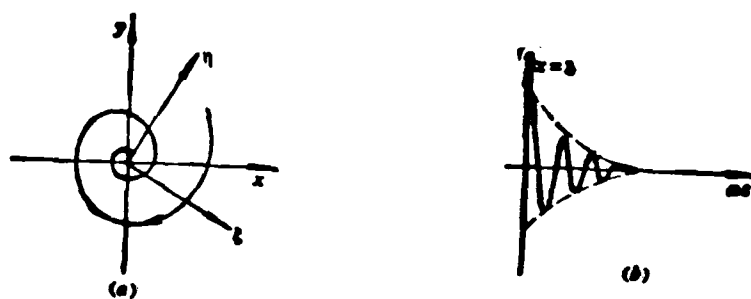


Figure 2.4 A Stable Focus

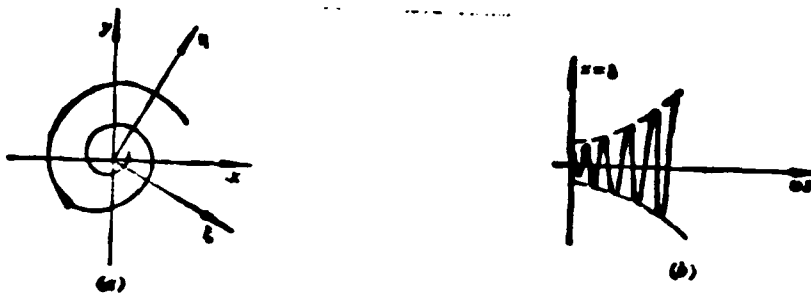


Figure 2.5 An Unstable Focus

A limit cycle is an isolated closed trajectory. A stable /65  
limit cycle represents the self-oscillation of the projectile.  
Because there is only one limit cycle, the projectile is always  
excited to approach the same steady vibration regardless of the  
initial perturbation. [See Figure 2.2 (b), (c)]

$$\delta = R_p \cos(\omega s + \varphi_0) \quad (2.1)$$

In this case, the motion of the projectile has a stable  
trajectory. Because  $k_{z_2} = 0$  and  $\dot{\varphi} = 0$ , the frequency of the steady  
vibration is identical to that of the linear oscillator. The  
formation of the steady vibration and the magnitude of the  
amplitude can also be derived based on equation (1.12) by direct  
integration and by letting  $s \rightarrow \infty$ . (The derivation is omitted.)

Mathematically, when  $k_{z_2} = 0$ ,  $b_0 < 0$  and  $b_2 > 0$ , equation (1.1)  
happens to be the van der pol equation. It had long been proven  
mathematically that the van der Pol equation has a stable limit  
cycle. This limit cycle, however, it not a circle. The precise

limit cycle with a medium damp ( $2b_0/k_{z_0} \approx 1.0$ ) is shown as curve  $L_p$  in Figure 2.6[2],[4],[6].

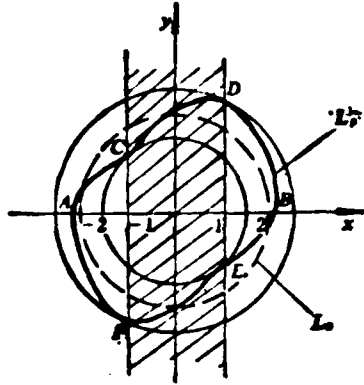


Figure 2.6 Precise Limit Cycle  $L_p$  and Approximate Limit Cycle  $L_a$  (in unit of  $\sqrt{-b_0/b_2}P$ )

From the energy point of view we can further comprehend the self-oscillation of the projectile as represented by the limit cycle. Let us multiply both sides of equation (1.1) by  $\delta'$ , we get

$$\frac{d(T+V)}{ds} = -2(b_0 + b_2\delta^2)(\delta')^2 \quad (2.2)$$

where  $T = 1/2(\delta')^2$  represents the angular kinetic energy of the projectile and  $V = -1/2 [k_{z_0}\delta^2 + (k_{z_2}/2)\delta^4]$  represents the potential energy of the static moment.

$T + V$  represents the total energy of the angular motion. Based on equation (2.2) we can see that the angular energy increases

when  $\delta^2 < -b_0/b_2$ . The total energy decreases when  $\delta^2 > -b_0/b_2$ . If the amount of energy absorbed by the projectile in a cycle happens to replenish the energy consumed, then the projectile can maintain a steady vibration. The self-oscillation represented by the motion of the phase point along the limit cycle has such characteristics.

Because the square of the phase radius,  $R^2 = \delta^2 + (\delta' / \sqrt{k_{z_0}})^2$ , is proportional to the total energy when  $k_{z_2} = 0$ , therefore, the limit cycle radius in the region  $|\delta| < \sqrt{-b_0/b_2}$  (shaded area in the figure) gradually increases on the precise limit cycle  $L_p$  shown in Figure 2.6. Outside this region, the radius is gradually decreasing because the energy lost in the segments DE and FC is compensated by that gained in segments EF and CD. Therefore, the projectile can stay in steady oscillation.

Self-oscillation occurs when  $b_0 < 0$  and  $b_2 > 0$  because  $b_0 < 0$  means that the damp is negative at small attack angles. In this case there is energy input in the angular motion. The damp is positive at large angles of attack ( $b_2 \delta^2 > |b_0|$ ) and it consumes energy in the angular motion. The energy input in the angular motion is obviously converted from the translational kinetic energy of the projectile. This can be qualitatively explained by the expression of  $b_0$ . Because  $b_0$  contains a term  $(b_z + g \sin \theta / v^2) < 0$  (rising arc), it acts as a negative damp which causes the angular energy to increase. Based on the equation of motion of the center of mass,  $(dv/dt) = -b_z v^2 - g \sin \theta$ , we know that that term also makes the velocity of the center of mass and the translational kinetic energy  $1/2 m v^2$  decrease. Thus, the amount of energy

gained angularly is converted from the translational kinetic energy. In terms of the physical concept, the total energy of the projectile only includes the angular and translational kinetic energy. Thus, an increase in the angular kinetic energy can only be converted from the translational kinetic energy. For projectiles of certain aerodynamic profiles, the drag is particularly large because the equatorial moment and lift are small. In this case, the situation  $b_0 < 0$  may occur. Consequently, it self-oscillates. This is a special motion different from any linear motion.

We notice that mechanical energy conservation exists on the /66 approximate limit cycle (the circle  $R=R_p$ ). Consequently, it is not possible to display the variation of energy in the angular motion. This is obviously an error caused by the first approximation. From Figure 2.6 we can also see that the approximate limit cycle is more or less the average position of the precise limit cycle. The smaller  $b_0$  and  $b_2$  become, the closer the precise limit cycle approaches a circle. Thus, the smaller the effect of this error becomes.

(2) When  $b_0 > 0$  and  $b_2 < 0$ , based on a smaller analysis we know that the  $R=R_p$  circle in the fixed phase plane is an unstable limit cycle. The origin is a stable focus. Therefore, when the initial condition is  $0 < R < R_p$ ,  $\dot{R} < 0$ , the phase point is damped to the origin [see Figure 2.3 (a)]. The corresponding oscillation curve is shown in Figure 2.3(b). The motion of the projectile is stabilizing. When  $R > R_p$ ,  $\dot{R} > 0$ . The phase point is far away from

the limit cycle. The corresponding oscillation curve is shown in Figure 2.3 (c). The motion of the projectile is unstable.

(3) When  $b_0 > 0$  and  $b_2 < 0$ , only the origin is a stable nodal point. In addition, we always have  $\dot{R} < 0$ . Hence, the motion of the projectile is stable (See Figure 2.4).

(4) When  $b_0 < 0$ ,  $b_2 < 0$ , we always have  $\dot{R} > 0$ . Hence, the motion of the projectile is unstable.

Finally, let us discuss the case that  $k_{z_2} \neq 0$ . From equation (1.13) we know that  $\dot{\phi} \neq 0$ . The phase point must also rotate around the origin in the dynamic phase plane. The variation of the phase radius, however, is still equation (1.12). Therefore, the circle  $R_p = 2\sqrt{-b_0/b_2}$  is still the limit cycle. Other phase trajectories are elongated clockwise ( $k_{z_2} < 0$ ) or squeezed counterclockwise ( $k_{z_2} > 0$ ). Correspondingly, the vibration frequency of the projectile will be higher and lower than the frequency of the linear oscillator. Furthermore, it causes the frequency to be related to the amplitude. However, the stability of the motion is not affected.

### 3. Conclusions and Analysis

Based on the above analysis we know that the motion of a stable static finned projectile with weak non-linear static moment and small damp is becoming stable when (1)  $b_0 > 0$ ,  $b_2 > 0$ . (2) When  $b_0 < 0$ ,  $b_2 < 0$ , the motion of the projectile is unstable. (3) When  $b_0 < 0$  and  $b_2 > 0$ , there is a stable limit cycle (the circle  $R=R_p$ ) in the phase plane. In this case, it is possible to create a self-oscillation at an amplitude  $R_p$ . (4) When  $b_0 > 0$  and

$b_2 < 0$ , there is an unstable limit cycle (the circle  $R=R_p$ ) in the phase plane. If  $R > R_p$ , then it may be in unstable motion.

In the design experiment of finned projectiles, the second case described must undoubtedly be eliminated. The linear theory, however, neglects the non-linear damp term  $b_2$ . It believes that as long as  $b_0 > 0$ , a stable static finned projectile must be in stable motion. Most of the finned projectiles were also thus designed. In reality, however,  $b_2$  does exist. The following two situations may appear. One is that  $b_2 > 0$  over the entire trajectory so that non-linear stability coincides with linear stability. Thus, the effect of  $b_2$  is masked. The other is  $b_2 < 0$  over the entire trajectory or on a portion of the trajectory to produce an unstable limit cycle in the phase plane. When the initial disturbance is excessive or when encountering a gust in flight, the phase point may jump out of the unstable limit cycle, i.e.,  $R = \sqrt{\delta^2 + (\delta'/\omega)^2} > R_p$ . Hence, the motion becomes unstable, leading to the fortuitous fall of projectiles. Because the probability of having an extraordinarily large initial disturbance or gust is very small, especially on trajectories where  $b_2 < 0$  occurs only in certain sections, the fall of projectile due to the unstable limit cycle is accidental.

As for fallen finned projectiles, we should inspect their  $b_2$  values at various Mach numbers. If  $b_2 < 0$ , we should alter the aerodynamic profile and layout to make  $b_2 > 0$  or to minimize  $|b_2|$ . Because the radius of the unstable limit cycle gets larger with smaller  $|b_2|$ , the range of initial conditions which satisfies

that of a stable motion is also getting wider. When  $|b_2| \rightarrow 0$ ,  $R_p \rightarrow \infty$ . Then, any initial condition cannot cause unstable motion.

In addition, certain odd-shaped projectiles with small equatorial moment and lift and large drag may encounter the situation that  $b_0 < 0$  and  $b_2 > 0$  at some Mach numbers. In this case the projectile will remain in self-oscillation at certain attack angles. If the amplitude of this oscillation is too large, the projectile is flying at a large angle of attack over a long period of time, leading to decreasing range and deteriorating degree of concentration. Therefore, the aerodynamic profiles of this type of projectiles should also be changed. The value  $b_2$  /67 should be increased to reduce the radius of the stable limit cycle in order to allow the projectile to fly at a small angle of attack.

#### References

- [1] Pu Fa, ((External Ballistics)), Beijing, Defense Industry Publishing Co, 1980.
- [2] Bogelubofu (author), Jin Fuling (translator), ((Approximation Methods in Non-linear Vibration Theory)), 1963.
- [3] Babakefu (author), Cai Chenguen, ((Vibration Theory)), 1963.
- [4] Department of Mathematics, Fudan University, ((Vibration Theory)), 1960.
- [5] Andelunuofu (author), Gao Weibin (translator), ((Vibration Theory)), 1981.
- [6] Murphy, C.H. Free Flight Motion of Symmetric Missiles, BRL 1216, 1963.

### Abstract

In this paper, non-linear motion of finned projectile with small damp and weak non-linear static moment has been studied. Using the stable limit cycle, it has been explained that some finned projectiles probably produce self-oscillation.

Using the unstable limit cycle, the causes of the fortuitous fall of some finned projectiles have been discussed. In order to decrease the influence of the self-oscillation and eliminate the fortuitous fall of the finned projectile, the non-linear part of the damping moment and lift must be increased, and the non-linear part of the drag must be decreased.

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